limit of 1.00 for a wall ratio of 2.0 and 1.315 for a cylinder where $R = 9.0$. This is just another way of saying that if the wall ratio of a cylinder is increased from 2 to 9, the elastic-limit pressure increases 31.5%. Furthermore, still referring to Table V, a cylinder having a wall ratio of 9 and a side hole ratio of 5 exhibits an elastic-limit pressure equal to 64.5% of that exhibited by a plain monobloc cylinder having a wall ratio of 2.0.

The real value of having more precise values for stress concentration factors in cylinders with side holes can be illustrated by noting the relative elastic-limit values in Table V corresponding to the last column marked "Variable ($K = 6$)." In many conventional designs it has been customary to use K values as high as 6.0, regardless of the side hole size, simply because the true value has not been known. When such a large factor is used, for example, the calculated elastic-limit pressure for a cylinder having a wall ratio of 4.0 is only 29.6% of that exhibited by a plain monobloc cylinder having a wall ratio of 2.0, regardless of the hole size (R_s value). Now, if R_s is taken into account, Table V shows for the same cylinder having a wall ratio of 4.0, that for an R_s of 5.0, the elastic-limit pressure is 60.5% of that exhibited by the plain monobloc of wall ratio $R = 2.0$. In other words, a factor of about 100% is involved and when the true value of K is used overdesign is avoided. Safety factors are always used; however, if the safety factor is 2, the equipment should be proportioned to give this factor of safety and not a factor of 4, which may occur if the proper values of K are not used.

Example 3. Use of Elliptic Side Holes. When a circular side hole is placed in a cylinder, the maximum stress concentration occurs in the hoop direction at the side hole–bore interface. This K factor can be reduced by making the side hole elliptic in shape; however, it is important not to introduce a K factor at the ends of the major axis of the ellipse which, when applied to the longitudinal stress, would create a situation worse than that obtained in the hoop direction for a circular side hole.

The limiting case for a single small elliptic side hole will now be considered for cylinders with both open and closed ends. In the closed-end cylinder, the total effective hoop stress at the ends of the minor ellipse axis is given by the sum of Equations 2 and 4,

$$\sigma_h = \frac{2p_oR^2}{R^2 - 1}(1 + 2\,b/a) - \frac{p_oR^2}{R^2 - 1} \quad (36)$$

The total effective longitudinal stress at the ends of the major ellipse axis is given by the sum of Equations 3 and 5,

$$\sigma_z = \frac{p_oR^2}{R^2 - 1}(1 + 2\,a/b) - \frac{2p_oR^2}{R^2 - 1} \quad (37)$$

From Equations 9 and 36, the stress concentration factor at the end of the minor axis is calculated to be

$$K_b = \frac{1 + 4^{\,b/a}}{2} \quad (38)$$

and by Equations 9 and 37, the stress concentration factor at the end of the major axis is

$$K_a = 2\,\frac{a}{b} - 1 \quad (39)$$

To determine the limiting ellipse axis ratio it is necessary to equate the equivalent stresses, as given by Equation 28, at the ends of the two axes; thus,

$$(\sigma_o)_a{}^2 = (\sigma_o)_b{}^2 = (K_b\sigma_h)^2 + 2p(K_b\sigma_h) = (K_a\sigma_z)^2 + 2p(K_a\sigma_z) \quad (40)$$

For a closed-end cylinder, $\sigma_h = \sigma_z(R^2 + 1)$; therefore, by substituting this value of σ_h in Equation 40 along with values of K_b and K_a given by Equations 38 and 39, the limiting axis ratio, a/b, may be determined as a function of the wall ratio, R, as shown in Table VI.

If the ends of the cylinder are open, the longitudinal stress (Equation 11) becomes

$$\sigma_z = p_o \quad (41)$$

and Equations 36 and 27 become, respectively,

$$\sigma_h = \frac{2p_oR^2}{R^2 - 1}\left(1 + 2\,\frac{b}{a}\right) - p_o \quad (42)$$

and

$$\sigma_z = p_o(1 + 2\,a/b) - \frac{2p_oR^2}{R^2 - 1} \quad (43)$$

Similarly, Equations 38 and 39 become, respectively

$$K_b = (1 + 2\,b/a) - \frac{(R^2 - 1)}{2R^2} \quad (44)$$

and

$$K_a = (1 + 2\,a/b) - \frac{2R^2}{R^2 - 1} \quad (45)$$

Now, as before, by equating the equivalent stresses $(\sigma_o)_a$ and $(\sigma_o)_b$, it can be shown that there is no limiting axis ratio; in other words, for an open-end cylinder under internal pressure yielding will always initiate at the ends of the minor axis. The end condition of the cylinder has a large effect on the stress-concentrating effect of side holes, whereas in plain cylinders without side holes the effect is almost negligible.

Table V. Elastic-Limit Pressures for Closed-End Cylinders

Wall Ratio, R	Relative Elastic Limit for Side Hole Ratios of ∞	5	1	Variable ($K = 6$)
2	1.00	0.447	0.473	0.210
3	1.185	0.560	0.575	0.285
4	1.250	0.605	0.625	0.296
9	1.315	0.645	0.670	0.322

Table VI. Limiting Values of Axis Ratio for Elliptic Side Hole in Closed-End Cylinder

Wall Ratio, R	Axis Ratio, a/b
1.5	2.57
2.0	3.28
2.5	4.09
3.0	5.00
3.5	6.02
4.0	7.13
5.0	9.68
10.0	29.21

Intermediate axis ratios between 1 and the critical values can be used to reduce stress concentration effects in cylinders. For many cylinders such ellipses can be obtained by tangential drilling of the side hole, using a cylindrical drill.

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